Decoherence of Hydrodynamic Histories: A Simple Spin Model

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ABSTRACT: In the context of the decoherent histories approach to the quantum mechanics of closed systems, Gell-Mann and Hartle have argued that the variables typically characterizing the quasiclassical domain of a large complex system are the integrals over small volumes of locally conserved densities – hydrodynamic variables. The aim of this paper is to exhibit some simple models in which approximate decoherence arises as a result of local conservation. We derive a formula which shows the explicit connection between local conservation and approximate decoherence. We then consider a class of models consisting of a large number of weakly interacting components, in which the projections onto local densities may be decomposed into projections onto one of two alternatives of the individual components. The main example we consider is a one-dimensional chain of locally coupled spins, and the projections are onto the total spin in a subsection of the chain. We compute the decoherence functional for histories of local densities, in the limit when the number of components is very large. We find that decoherence requires two things: the smearing volumes must be sufficiently large to ensure approximate conservation, and the local densities must be partitioned into sufficiently large ranges to ensure protection against quantum fluctuations.

1. INTRODUCTION

One of the primary aims of quantum cosmology is to understand how the universe is classical to a very high degree of precision, given the hypothesis that it is described at the most fundamental level by quantum theory [1,2,3]. Mathematically, this aim translates into the question of why, on sufficiently large scales, quantum theory admits an emergent description of the universe involving only a small number of dynamical variables obeying an approximately closed set of deterministic evolution equations.

Whilst many approaches to this question concentrate on the positions and momenta of point particles, these dynamical variables are special cases of a more general description of the physical world in terms of local densities: number density, momentum density, energy density, charge density, *etc.* The question of emergent classicality then consists first, of understanding why these variables enjoy such a distinguished role, and second, of deriving the familiar hydrodynamic equations for these densities.

The decoherent histories approach to quantum theory is a recently developed formulation of quantum theory which is particularly suited to this problem [4,5,6,7,8,9]. In brief, the approach permits predictions to be made in genuinely closed systems, such as the entire universe, without relying on notions of measurement or on an external classical domain. The goal of the approach is to assign probabilities to the histories of a closed quantum system. It is this feature that makes it particularly useful for discussing emergent classicality. This is because to assert in quantum theory that a certain variable approximately satisfies a deterministic evolution equation involves computing the probability for a time-ordered sequence of values of that variable, *i.e.*, for a history. It is then necessary to show that the probability for that history is strongly peaked about the sequence of values corresponding to the evolution equation.

This paper constitutes the beginnings of a general application of the decoherent histories approach to the question of deriving hydrodynamic equations.

We begin with a very brief review of the decoherent histories approach [4,5,6,7,8,9,10], followed by a general discussion of the issue of emergent classicality for hydrodynamic variables. A more detailed statement of the particular models considered in

this paper may be found at the end of this section.

The models and technical results presented here are admittedly simple, perhaps to the extent of being overshadowed by the very broad discussion of emergent classicality given below. This paper is, however, a small first step in a possibly extensive programme. We have therefore chosen to give a general sketch of that programme, to indicate the direction of future research and to provide the general context in which the possible significance of our modest results may be understood.

1(A). The Decoherent Histories Approach to Quantum Theory

A quantum-mechanical history is a sequence of propositions at a succession of times. Propositions at a fixed moment of time are represented by sets of projections operators, $\{P_\alpha\}$. They are exhaustive and exclusive,

$$\sum_\alpha P_\alpha = 1, \qquad P_\alpha P_\beta = \delta_{\alpha\beta}\ P_\alpha \tag{1.1}$$

A projector is said to be fine-grained if it is of the form $|\alpha\rangle\langle\alpha|$, where $\{|\alpha\rangle\}$ are a complete set of states. Otherwise it is said to be coarse-grained. A single quantum-mechanical history is characterized by a string of time-dependent projections, $P^1_{\alpha_1}(t_1), \cdots, P^n_{\alpha_n}(t_n)$, together with an initial state ρ. The time-dependent projections are related to the time-independent ones by

$$P^k_{\alpha_k}(t_k) = e^{iH(t_k - t_0)} P^k_{\alpha_k} e^{-iH(t_k - t_0)} \tag{1.2}$$

where H is the Hamiltonian. The candidate probability for such histories is

$$p(\alpha_1, \alpha_2, \cdots \alpha_n) = \mathrm{Tr}\left(P^n_{\alpha_n}(t_n) \cdots P^1_{\alpha_1}(t_1) \rho P^1_{\alpha_1}(t_1) \cdots P^n_{\alpha_n}(t_n) \right) \tag{1.3}$$

To be a true probability, Eq.(1.3) must satisfy the probability sum rules. That is, it must be such that the probability for each coarser-grained history is the sum of the probabilities for the constituent finer-grained histories. Coarser-grained histories may be constructed, for example, by summing the projections at each moment of time:

$$\bar{P}_{\bar{\alpha}} = \sum_{\alpha \in \bar{\alpha}} P_\alpha \tag{1.4}$$

The probability sum rules are generally not satisfied due to quantum interference.

Sets of histories which do not suffer interference, and hence for which the sum rules are satisfed may be found using the decoherence functional,

$$D(\underline{\alpha},\underline{\alpha}') = \mathrm{Tr}\left(P^n_{\alpha_n}(t_n)\cdots P^1_{\alpha_1}(t_1)\rho P^1_{\alpha'_1}(t_1)\cdots P^n_{\alpha'_n}(t_n)\right) \tag{1.5}$$

Here $\underline{\alpha}$ denotes the string $\alpha_1,\alpha_2,\cdots\alpha_n$. Intuitively, the decoherence functional measures the amount of interference between pairs of histories. It may be shown that the probability sum rules are satisfied for all coarse-grainings if and only if

$$ReD(\underline{\alpha},\underline{\alpha}') = 0 \tag{1.6}$$

for all distinct pairs of histories $\underline{\alpha},\underline{\alpha}'$ [7]. Such sets of histories are said to be *consistent* (or *weakly decoherent*).

The consistency condition (1.6) is usually (but not always) satisfied only for coarse-grained histories. When sets of histories satisfy the consistency condition (1.6) as a result of coarse-graining, they often satisfy, in addition, the stronger condition of *decoherence*,

$$D(\underline{\alpha},\underline{\alpha}') = 0, \quad for \quad \underline{\alpha}\neq\underline{\alpha}' \tag{1.7}$$

(sometimes classified as *medium* decoherence, with yet more stringent criteria for *strong* decoherence [6]). Physically, decoherence is intimately related to the existence of records about the system somewhere in the universe [5].

In most cases of interest decoherence is only approximate, so measures of approximate decoherence are required. We say that the degree of decoherence is of order ϵ if the probability sum rules are violated only up to terms of order ϵ times the probabilities themselves [10]. It may be shown that, under certain conditions on the distribution of off-diagonal terms in the decoherence functional, decoherence to order ϵ will be satisfied if

$$|D(\underline{\alpha},\underline{\alpha}')|^2 \;<\; \epsilon^2\; D(\underline{\alpha},\underline{\alpha})D(\underline{\alpha}',\underline{\alpha}'). \tag{1.8}$$

1(B). Emergent Quasiclassicality

We would like to use the decoherent histories approach to demonstrate the emergence of an approximately classical world from an underlying quantum one,

together with the quantum fluctuations about it described by the standard Copenhagen quantum mechanics of measured subsystems. Such a state of affairs is referred to as a quasiclassical domain [4,5,6]. In more technical terms, a quasiclassical domain consists of a decoherent set of histories, characterized largely by the same types of variables at different times, and whose probabilities are peaked about deterministic evolution equations for the variables characterizing the histories.

The histories should, moreover, be *maximally refined* with respect to a specified degree of approximate decoherence. That is, one specifies a decoherence factor ϵ in the approximate decoherence condition discussed above. This should, for example, be chosen so that the probabilities are defined to a precision far beyond any conceivable test. Then, the histories should be fine-grained (*e.g.*, by reducing the widths of the projections) to the point that further fine-graining would lead to violation of the specified degree of approximate decoherence. The resulting set of histories are then called maximally refined. The reason for maximally refining the histories is to reduce as much as possible any apparent subjective element in the choice of coarse-graining.

Given the Hamiltonian and initial state of the system, one's task is to compute the decoherence functional for various different choices of histories, and see which ones lead to quasiclassical behaviour. We expect this to be a formidable task, and there is no reason to believe that there will be a unique answer. Generally, one might expect that there will be a hierarchy of variables, similar to the BBGKY hierarchy [11].

Many previous discussions [5,10,12,13,14] of emergent classicality concern systems in which there is a natural separation of the total closed sytem into "system" and "environment", and this separation is the source of the coarse-graining required for decoherence.† A generic closed system, however, will usually not have such a separation, and it is one of the strengths of the decoherent histories approach that it does not rely on the existence of one. Certain variables will, however, be distinguished by the existence of conservation laws for total energy, momentum, charge, particle number, *etc.* Associated with such conservation laws are local conservation

† More precisely, by separation into system and environment, we mean that the total Hilbert space for the closed system may be written as a tensor product of the system and environment Hilbert spaces. Some authors appear to use this expression to mean something more general.

laws of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{1.9}$$

The candidate quasiclassical variables are then

$$Q_V = \int_V d^3x \ \rho(\mathbf{x}) \tag{1.10}$$

These are the local densities discussed above. If the volume V becomes infinite, Q_V will be an exactly conserved quantity. In quantum mechanics it will commute with the Hamiltonian, and, as is easily seen, histories of Q_V's will decohere exactly [15]

If the volume is finite but large compared to the microscopic scale, Q_V will be slowly varying compared to other dynamical variables. This is because the local conservation law (1.9) permits Q_V to change only by redistribution, which is limited by the rate at which the locally conserved quantity can flow out of the volume. Because these quantities are slowly varying, histories of them should approximately decohere.

More precisely, when the volume V is infinite in size, the decoherence functional for histories of the variables (1.10) will be exactly diagonal. As V shrinks from infinite size, the decoherence functional will develop off-diagonal terms. However, one would not expect the off-diagonal terms to grow appreciably until V approaches the length scales characteristic of the system at hand. These length scales can depend only on the initial state and on the Hamiltonian. The length scale associated with the initial state could in principle take any value, hence some restriction on the initial state will be necessary. The length scales associated with Hamiltonians of the type we are typically interested in will generally be very small. The various physically relavent scales, are, for example, the mean free path between molecular collisions, or the length scale of intermolecular forces. Therefore, for some class of initial states (to be determined), one would expect the off-diagonal terms of the decoherence functional to remain small until V shrinks down to the microscopic scale.

Hence, for a suitable class of initial states, we expect the variables (1.10) to be approximately decoherent as a consequence of their association with conservation laws, which are in turn connected with the global symmetries of the system.

Given decoherence, we may then examine the probabilities for histories of hydrodynamic variables, and ask whether they are peaked about the expected hydro-

dynamic equations. Derivations of the hydrodynamics equations from an underlying microscopic quantum theory have certainly been carried out before (see Ref.[16] for example). These derivations have shown that the *expectation values* of the local densities Q_V, in a local equilibrium state, evolve in approximate accordance with the hydrodynamic equations. The derivation contemplated here, however, is considerably more general (although undoubtedly related in some way). In the decoherent histories approach we would like to show that the probabilities for histories of imprecisely specified values of the local densities, for a more general class of initial states, are peaked about the hydrodynamic equations. This paper is only a first step in that direction; but it is important to keep the ultimate goal in mind.

The connection between the earlier derivations of the hydrodynamic equations and the histories derivation contemplated here is analagous to the connection between the density operator and the decoherent histories approaches to emergent classicality of quantum Brownian motion models. In the density operator approach, it was shown that an initial Gaussian wave packet followed an approximately classical path, with dissipation, and with fluctuations due to classical and quantum noise [13,12,17]. In the decoherent histories approach, it was shown that for a wide variety of initial states, the probabilities for histories of position samplings is strongly peaked about classical evolution, with dissipation, with a width of peaking depending on classical and quantum noise [5,10].

1(C). This Paper

The programme sketched above is clearly a very extensive one. Although clear in principle how to proceed, it is very difficult to carry out in practice. In principle, since the operators are known, one may compute their spectrum, from which the projectors onto ranges of the spectrum may be derived. The time evolution of those projectors may be deduced and inserted in the decoherence functional. In practice, this is very difficult and it seems likely that new mathematical techniques will be needed.

It is unlikely, for example, that the path integral techniques so successful in the study of quantum Brownian motion models will be effective here [5,10]. In the quantum Brownian motion models, the interesting variables are the position or momentum of a distinguished particle, and projections onto these variables are easily

implemented as restrictions on the paths in a sum over paths. The hydrodyanmic variables considered here, however, are non-trivial functions of positions and momenta, and projections onto ranges of their spectra cannot in general be expressed in terms of restrictions in a sum over paths in phase space.

The quantum Brownian motion models also made heavy use of the influence functional method [18,19]. Again this is inapplicable, because it relies on being able to explicitly integrate out the "environment" (*i.e.*, the variables ignored in the coarse-graining procedure), which cannot be done here.

To make a start on the general problem what is required is some very simple models which retain the essential features of approximate decoherence through approximate conservation, yet are simple enough to be solvable. In this paper we will present some models which are of this type.

In Section II, we derive a formula expressing the general connection between approximate conservation and approximate decoherence. We will not use this formula explicitly in this paper, although we present it to flesh out the heuristic arguments given above, and because it may be of practical use elsewhere.

In Section III, we consider a system consisting of a large number N of weakly interacting components in which the individual components are described by just two alternatives. An example is a large number of non-interacting particles in a box divided into two sections, and the two alternatives for each individual component are that the particle is in the left or right section of the box. A crude hydrodynamic variable is then the number of particles in, say, the right hand section. The projection onto such a hydrodynamic variable can be given in terms of the projections onto the individual particles, and we derive a formula which gives this connection precisely. Another example is the spin system model we describe below. For this class of models, we compute the decoherence functional for histories characterized by projections onto densities of large collections of particles. We thus compute the degree of decoherence.

The calculation of Section III is principally concerned with the combinatoric aspects of large collections of particles. The feature of decoherence through approximate conservation, with which we are ultimate concerned, is to be found in the decoherence functionals for the individual components. Hence we need to compute such a decoherence functional explicitly. This is carried out for a spin model in

Section IV.

The model consists of a long chain of M locally coupled spins, and the Hamiltonian conserves the total spin. We employ the simplest coarse-graining possible, which is to divide the chain into two pieces, of lengths M_1 and M_2 (so $M_1 + M_2 = M$), and project onto the amount of spin into one section of the chain.

The ground state of the system is the state in which all the spins point up. The first excited states, the so-called spin waves, consist of superpositions of particle-like states in which one of the spins is down. The higher excited states are more difficult to construct because they involve interactions between spin waves, but they may be approximated by suitable combinations of the first excited states in the limit that the coupling is weak. In this approximation the system is therefore not unlike a collection of weakly interacting particles. We thus show that the decoherence functional for projections onto the number of, say, down spins in one section may be approximated by a product of decoherence functionals for the "one particle" states, in which the projections ask in which side is the down spin. This means that, in this approximation, our spin model is a system of the type considered in Section III.

We calculate the decoherence functional for the individual components numerically. It is approximately diagonal, but not surprisingly, the degree of decoherence is not very good, being no better than would be expected from the overlap of a random pair of states in a large Hilbert space. Its significant feature is the way the degree of decoherence changes as M_1 and M_2 are changed. We use this result, together with the formula derived in Section III, to compute the degree of decoherence for projections onto ranges of spin density.

We summarize and conclude in Section V.

2. APPROXIMATE CONSERVATION AND APPROXIMATE DECOHERENCE

In this section we briefly outline the connection between approximate conservation and approximate decoherence.

As in Section I, let Q denote a local density averaged over a volume V:

$$Q = \int_V d^3x \ \rho(\mathbf{x}) \tag{2.1}$$

By integrating the conservation equation for ρ, Eq.(1.9), over a time interval $[0, t]$ and over the spatial volume V, it is readily shown that

$$Q_t = Q - \int_0^t dt' \int_{\partial V} d^2x \ \mathbf{n} \cdot \mathbf{j}(\mathbf{x}, t') \tag{2.2}$$

where ∂V is the boundary of V. This relation will also hold in the quantum theory for suitably ordered operators.

Now consider the decoherence functional for histories characterized by projections onto these operators at two moments of time:

$$D(\alpha_1, \alpha_2|\alpha_1', \alpha_2) = \mathrm{Tr}\left(P_{\alpha_2} e^{-iHt} P_{\alpha_1} \rho P_{\alpha_1'} e^{iHt}\right) \tag{2.3}$$

This may be written

$$D(\alpha_1, \alpha_2|\alpha_1', \alpha_2) = \sum_{m,n} \langle m|P_{\alpha_2} e^{-iHt} P_{\alpha_1}|n\rangle \ \rho_{nm} \langle m|P_{\alpha_2} e^{-iHt} P_{\alpha_1'}|n\rangle^* \tag{2.4}$$

where $|n\rangle$ denotes a complete set of states, and $\rho_{nm} = \langle n|\rho|m\rangle$. The degree of decoherence will therefore depend on the size of the amplitudes,

$$\mathcal{A}_{mn}(\alpha_1, \alpha_2) = \langle m|P_{\alpha_2} e^{-iHt} P_{\alpha_1}|n\rangle \tag{2.5}$$

Projections onto exactly conserved quantities will commute with H, and $\mathcal{A}_{mn}(\alpha_1, \alpha_2)$ will be exactly zero unless $\alpha_2 = \alpha_1$, hence the decoherence functional will be exactly diagonal. We are interested, however, in the case in which the projections commute only approximately with the Hamiltonian.

Suppose the spectrum of the operator Q is continuous. Then projections onto precisely specified values of its eigenvalues may be constructed using the delta-function:

$$P_\alpha = \delta(Q - \alpha) \tag{2.6}$$

(These are not fine-grained projections, since the eigenvalues will typically be highly degenerate). Projections onto imprecisely specified ranges of eigenvalues are readily constructed by summing over ranges of α, although we will not do this explicitly here.

The amplitudes $\mathcal{A}_{mn}(\alpha_1, \alpha_2)$ may now be written

$$\begin{aligned}\mathcal{A}_{mn}(\alpha_1, \alpha_2) &= \langle m|\delta(Q-\alpha_2)\ e^{iHt}\ \delta(Q-\alpha_1)|n\rangle \\ &= \langle m|\delta(Q-\alpha_2)\ e^{-i\epsilon Q}\ e^{i\epsilon Q}\ e^{iHt}\ \delta(Q-\alpha_1)|n\rangle \end{aligned} \tag{2.7}$$

where ϵ is an arbitrary parameter. Now we use the fact that

$$e^{i\epsilon Q_t} = e^{iHt}\ e^{i\epsilon Q}\ e^{-iHt} \tag{2.8}$$

where $Q_t = e^{iHt} Q e^{-iHt}$. Hence we have

$$\mathcal{A}_{mn}(\alpha_1, \alpha_2) = \langle m|\delta(Q-\alpha_2)\ e^{-i\epsilon Q}\ e^{-iHt}\ e^{i\epsilon Q_t}\ \delta(Q-\alpha_1)|n\rangle \tag{2.9}$$

Q_t is also given by (2.2), so inserting (2.2), expanding to first order in ϵ, using the fact that $Q\delta(Q-\alpha) = \alpha\delta(Q-\alpha)$, and rearranging, we get

$$\mathcal{A}_{mn}(\alpha_1, \alpha_2) = -\frac{1}{(\alpha_2-\alpha_1)} \int_0^t dt' \int_{\partial V} d^2x\ \langle m|\delta(Q-\alpha_2)\ e^{iHt}\ \mathbf{n}\cdot\mathbf{j}(\mathbf{x}, t')\ \delta(Q-\alpha_1)|n\rangle \tag{2.10}$$

Eq.(2.10) is the main result of this section. Inserted in the decoherence functional Eq.(2.4), (2.10) yields the degree of decoherence as a function of the size of the smearing volume V. In particular, we see that decoherence becomes exact as the boundary ∂V of V goes to infinity, as it must, since the operators Q are then exactly conserved. Of course, the rate at which approximate decoherence approaches exact decoherence will depend on the states $|n\rangle$, $|m\rangle$, in accordance with our general expectations.

In the case of exact conservation the amplitude $\mathcal{A}_{mn}(\alpha_1, \alpha_2)$ is non-zero only for $\alpha_2 = \alpha_1$, hence the probability of α_2 given α_1 is 1 when $\alpha_2 = \alpha_1$, and zero otherwise. In the case of approximate conservation, however, one expects the situation to be a little more complicated. Given decoherence, the probability distribution for histories will not necessarily be peaked about $\alpha_2 = \alpha_1$. Rather, since we expect the probabilities for histories to indicate hydrodynamic equations, it will be peaked about values of α_1, α_2 consistent with some deterministic evolution equations. For this reason, it is perhaps better to speak of approximate determinism, rather than approximate conservation.

3. DECOHERENCE OF DENSITIES OF LARGE COLLECTIONS OF NON-INTERACTING PARTICLES

3(A). The System

Consider a system which consists of a very large number N of particles or components whose interactions are so weak that they may be neglected. We are interested in the case in which each component is described by just two alternatives at each moment of time, which may be represented by projections P and $\bar{P} = 1 - P$. An example is a collection of particles in a box divided into two sections, and the projections P and $\bar{P}$ then represent the propositions that the particle is in, respectively, the right-hand or left-hand section of the box. A version of this system involving spins will be described in the next section. We will refer to the alternatives represented by P and $\bar{P}$ as "yes" and "no", respectively.

Using these elementary projections onto the individual components of the system, projections onto densities of the whole system may be constructed. For a system of two particles, for example, the number of particles in the right-hand section of the box, may be 2, 1 or 0. These propositions are represented, respectively, by the projections,

$$P_2 = P \otimes P \tag{3.1}$$

$$P_1 = P \otimes \bar{P} + \bar{P} \otimes P \tag{3.2}$$

$$P_0 = \bar{P} \otimes \bar{P} \tag{3.3}$$

It is easy to see that these projections are mutually exclusive and exhaustive, as required.

For large systems, the projections onto densities rapidly become quite cumbersome. However, the following trick turns out to be extremely useful. We have the identity,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \; e^{-i\lambda n} e^{i\lambda m} = \delta_{nm} \tag{3.4}$$

where δ_{nm} is the Kronecker delta. The projection operator onto number density $\bar{n}$ in a system of N components is then given by

$$P_{\bar{n}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \; e^{-i\lambda\bar{n}} \; F_1(\lambda) \otimes F_2(\lambda) \cdots \otimes F_N(\lambda) \tag{3.5}$$

Here

$$F_k(\lambda) = e^{i\lambda} P^{(k)} + \bar{P}^{(k)} \tag{3.6}$$

where $P^{(k)}$ and $\bar{P}^{(k)}$ denote the projections P, $\bar{P}$ operating on component number k. What happens in Eq.(3.5) is that in the tensor product over the $F_k(\lambda)$'s, the projection onto number density n occurs with coefficient, $e^{i\lambda n}$, *i.e.*,

$$\begin{aligned} F_1(\lambda) \otimes F_2(\lambda) \cdots \otimes F_N(\lambda) =& \bar{P} \otimes \bar{P} \otimes \cdots \otimes \bar{P} \\ &+ e^{i\lambda} \left(P \otimes \bar{P} \otimes \cdots \otimes \bar{P} + \bar{P} \otimes P \otimes \cdots \otimes \bar{P} + \cdots \right) \\ &+ \cdots \\ =& \sum_{n=0}^{N} e^{i\lambda n} \; P_n \end{aligned} \tag{3.7}$$

The integration over λ then picks out only the term with coefficent $e^{i\lambda \bar{n}}$.

These projections are onto precise values of number density. Generally one is interested in imprecise values of number density, *i.e.*, whether the number density lies in a specified range, and these projections are obtained by summing over $\bar{n}$. We will consider this in more detail below.

3(B). The Decoherence Functional

We may now write down the decoherence functional for histories characterized by projections at two moments of time onto precise values of number density. It is,

$$\begin{aligned} D(\bar{n}_1, \bar{n}_2 | \bar{n}_1', \bar{n}_2) &= \mathrm{Tr} \left(P_{\bar{n}_2} e^{-iHt} P_{\bar{n}_1} \rho P_{\bar{n}_1'} e^{iHt} P_{\bar{n}_2} \right) \\ &= \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\lambda_1 \int_{-\pi}^{\pi} d\lambda_1' \int_{-\pi}^{\pi} d\lambda_2 \int_{-\pi}^{\pi} d\lambda_2' \; e^{-i\lambda_1 \bar{n}_1 + i\lambda_1' \bar{n}_1' - i\lambda_2 \bar{n}_2 + i\lambda_2' \bar{n}_2} \\ &\times \mathrm{Tr} \Big(F_1(\lambda_2) \otimes F_2(\lambda_2) \cdots \otimes F_N(\lambda_2) \; e^{-iH_T t} \; F_1(\lambda_1) \otimes F_2(\lambda_1) \cdots \otimes F_N(\lambda_1) \\ &\times \; \rho_T \; F_1^\dagger(\lambda_1') \otimes F_2^\dagger(\lambda_1') \cdots \otimes F_N^\dagger(\lambda_1') \; e^{iH_T t} \; F_1^\dagger(\lambda_2') \otimes F_2^\dagger(\lambda_2') \cdots \otimes F_N^\dagger(\lambda_2') \Big) \end{aligned} \tag{3.8}$$

Note that we can of course use the simplifying feature $(P_{\bar{n}_2})^2 = P_{\bar{n}_2}$ in the final projection, but it will become clear below why we have not done this.

To make further progress we make two simplifications. First, we assume that the interaction between component particles is negligible, so the Hamiltonian H_T

for the total system has the form

$$H_T = H \otimes 1 \cdots \otimes 1 + 1 \otimes H \cdots \otimes 1 + \cdots \tag{3.9}$$

Second, we assume that the intial density operator ρ_T for the total system factors, and that the density operator for each component system is the same:

$$\rho_T = \rho \otimes \rho \cdots \otimes \rho \tag{3.10}$$

The decoherence functional may now be written

$$\begin{aligned} D(\bar{n}_1, \bar{n}_2 | \bar{n}'_1, \bar{n}_2) = \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\lambda_1 \int_{-\pi}^{\pi} d\lambda'_1 \int_{-\pi}^{\pi} d\lambda_2 \int_{-\pi}^{\pi} d\lambda'_2 \; e^{-i\lambda_1 \bar{n}_1 + i\lambda'_1 \bar{n}'_1 - i\lambda_2 \bar{n}_2 + i\lambda'_2 \bar{n}_2} \\ \times \prod_{k=1}^{N} \mathrm{Tr} \left(F_k(\lambda_2) \; e^{-iHt} \; F_k(\lambda_1) \; \rho \; F_k^\dagger(\lambda'_1) \; e^{iHt} \; F_k^\dagger(\lambda'_2) \right) \end{aligned} \tag{3.11}$$

where the trace is now over the Hilbert space of the component system.

The last part of the integrand may be written

$$\begin{aligned} &\prod_{k=1}^{N} \mathrm{Tr} \left(F_k(\lambda_2) \; e^{-iHt} \;\; F_k(\lambda_1) \; \rho \; F_k^\dagger(\lambda'_1) \; e^{iHt} \; F_k^\dagger(\lambda'_2) \right) \\ &\qquad = \left[\mathrm{Tr} \left(F_k(\lambda_2) \; e^{-iHt} \; F_k(\lambda_1) \; \rho \; F_k^\dagger(\lambda'_1) \; e^{iHt} \; F_k^\dagger(\lambda'_2) \right) \right]^N \\ &\qquad = \Big[e^{i(\lambda_1 + \lambda_2 - \lambda'_1 - \lambda'_2)} p(y,y) + e^{i(\lambda_2 - \lambda'_2)} p(n,y) \\ &\qquad\quad + e^{i(\lambda_1 - \lambda'_1)} p(y,n) + p(n,n) \\ &\qquad\quad + e^{i(\lambda_1 + \lambda_2 - \lambda'_2)} D(y,y|n,y) + e^{i(\lambda_2 - \lambda'_1 - \lambda'_2)} D(n,y|y,y) \\ &\qquad\quad + e^{i\lambda_1} D(y,n|n,n) + e^{-i\lambda'_1} D(n,n|y,n) \Big]^N \end{aligned} \tag{3.12}$$

where $p(y,y)$, $D(y,y|n,y)$ *etc.* denote the probabilities and decoherence functionals for each component system, that is

$$p(y,y) = \mathrm{Tr} \left(P \; e^{-iHt} \; P \; \rho \; P \; e^{iHt} \right) \tag{3.13}$$

$$p(n,y) = \mathrm{Tr} \left(P \; e^{-iHt} \; \bar{P} \; \rho \; \bar{P} \; e^{iHt} \right) \tag{3.14}$$

$$p(y,n) = \mathrm{Tr} \left(\bar{P} \; e^{-iHt} \; P \; \rho \; P \; e^{iHt} \right) \tag{3.15}$$

$$p(n,n) = \mathrm{Tr} \left(\bar{P} \; e^{-iHt} \; \bar{P} \; \rho \; \bar{P} \; e^{iHt} \right) \tag{3.16}$$

$$D(y,y|n,y) = \mathrm{Tr} \left(P \; e^{-iHt} \; P \; \rho \; \bar{P} \; e^{iHt} \right) \tag{3.17}$$

$$D(n,y|y,y) = \mathrm{Tr} \left(P \; e^{-iHt} \; \bar{P} \; \rho \; P \; e^{iHt} \right) \tag{3.18}$$

$$D(y,n|n,n) = \mathrm{Tr} \left(\bar{P} \; e^{-iHt} \; P \; \rho \; \bar{P} \; e^{iHt} \right) \tag{3.19}$$

$$D(n,n|y,n) = \mathrm{Tr} \left(\bar{P} \; e^{-iHt} \; \bar{P} \; \rho \; P \; e^{iHt} \right) \tag{3.20}$$

The probabilities obey the simple relation

$$p(y,y) + p(n,y) + p(y,n) + p(n,n) = 1 \tag{3.21}$$

The off-diagonal terms of the decoherence functional satisfy

$$D(y,y|n,y) + D(y,n|n,n) = 0 \tag{3.22}$$

$$D(n,y|y,y) + D(n,n|y,n) = 0 \tag{3.23}$$

The latter relations are particular to decoherence functionals of histories characterized by projections at two moments of time. They are consistent with the general property that the sum of off-diagonal terms is zero. Also, since the second relation is the complex conjugate of the first, they imply that all the off-diagonal terms of the decoherence functional may be recovered from a single complex number, which may be taken, for example, to be $D(y,y|n,y)$.

3(C). Evaluation for Large N

The integral (3.11), with (3.12) inserted may be evaluate exactly by muliple use of the binomial expansion. This is carried out in Appendix A. More useful is an approximate evaluation for large N. To do this, note that

$$\begin{aligned}
&\left|\mathrm{Tr}\left(F_k(\lambda_2)\; e^{-iH_k t}\; F_k(\lambda_1)\;\; \rho_k\; F_k^\dagger(\lambda_1')\; e^{iH_k t}\; F_k^\dagger(\lambda_2')\right)\right|^2 \\
&\qquad \le \mathrm{Tr}\left(F_k(\lambda_2)\; e^{-iH_k t}\; F_k(\lambda_1)\; \rho_k\; F_k^\dagger(\lambda_1)\; e^{iH_k t}\; F_k^\dagger(\lambda_2)\right) \\
&\qquad\quad \times \mathrm{Tr}\left(F_k(\lambda_2')\; e^{-iH_k t}\; F_k(\lambda_1')\; \rho_k\; F_k^\dagger(\lambda_1')\; e^{iH_k t}\; F_k^\dagger(\lambda_2')\right) \\
&\qquad = 1
\end{aligned} \tag{3.24}$$

The first relation, the inequality, is an elementary generalization of the inequality

$$|D(\alpha,\alpha')|^2 \;\le\; D(\alpha,\alpha)\; D(\alpha',\alpha') \tag{3.25}$$

obeyed by the decoherence functional [10]. The second relation, equality with unity, follows from the fact that $F_k(\lambda)$ obeys the relation

$$F_k(\lambda) F_k^\dagger(\lambda) = 1 \tag{3.26}$$

as is easily shown.

The norm of the term raised to the power N when Eq.(3.12) is inserted in the integral (3.11) is therefore less than or equal to 1, with equality if and only if all the λ's are zero. It follows that for very large N, the integral over the λ's in Eq.(3.11) is dominated by values of the λ's close to zero, and the integral may therefore be evaluated by exanding about this point. (Note that we could not have used the inequality (3.24) if we had made the simplification $(P_{\bar{n}_2})^2 = P_{\bar{n}_2}$ in the final projection).

We now make the following change of variables:

$$\mu_1 = \frac{\lambda_1 + \lambda_1'}{2}, \quad \xi_1 = \frac{\lambda_1 - \lambda_1'}{2}, \quad \mu_2 = \frac{\lambda_2 + \lambda_2'}{2}, \quad \xi_2 = \frac{\lambda_2 - \lambda_2'}{2}. \tag{3.27}$$

The integral expression for the decoherence functional then becomes,

$$\begin{aligned} D(\bar{n}_1, \bar{n}_2|\bar{n}_1', \bar{n}_2) = \frac{1}{(2\pi)^4} \int d\mu_1 \; d\xi_1 \; d\mu_2 \; d\xi_2 \; e^{i\mu_1(\bar{n}_1' - \bar{n}_1) - i\xi_1(\bar{n}_1 + \bar{n}_1') - 2i\xi_2\bar{n}_2} \\ \times \Big[e^{2i(\xi_1+\xi_2)} p(y,y) + e^{2i\xi_2} p(n,y) + e^{2i\xi_1} p(y,n) + p(n,n) \\ + e^{i(\mu_1+\xi_1+2\xi_2)} D(y,y|n,y) + e^{i(-\mu_1+\xi_1+2\xi_2)} D(n,y|y,y) \\ + e^{i(\mu_1+\xi_1)} D(y,n|n,n) + e^{i(-\mu_1+\xi_1)} D(n,n|y,n) \Big]^N \end{aligned} \tag{3.28}$$

Since the integrand is independent of μ_2, the integral over μ_2 may be carried out. As explained above, for very large N, the integral over the remaining variables may then be carried out by expanding the integrand about $\mu_1 = \xi_1 = \xi_2 = 0$. To quadratic order, the term in brackets raised to the power N has the form,

$$\Big[1 + i\mathbf{x}\cdot\mathbf{v} - \mathbf{x}^T M \mathbf{x} \Big]^N \tag{3.29}$$

where $\mathbf{x}$ is the three-vector (μ_1, ξ_1, ξ_2), $\mathbf{v}$ is a three-vector, and M is a symmetric 3×3 matrix. The components of $\mathbf{v}$ may be read off from the following:

$$\begin{aligned} \mathbf{x}\cdot\mathbf{v} =& 2\left(p(y,y) + p(y,n)\right)\xi_1 \\ & + 2\left(p(y,y) + p(n,y) + D(y,y|n,y) + D(n,y|y,y)\right)\xi_2 \\ =& 2p_0\xi_1 + 2p_t\xi_2 \end{aligned} \tag{3.30}$$

where the properties of the decoherence functional (3.22), (3.23) have been used, and p_0 and p_t denote the probabilities of the alternative "yes" at times 0 and t respectively, *i.e*,

$$p_0 = \mathrm{Tr}\left(P\rho\right), \qquad p_t = \mathrm{Tr}\left(Pe^{iHt}\rho e^{-iHt}\right) \tag{3.31}$$

We also denote the probabilites of the "no" alternatives at times 0 and t by $\bar{p}_0$ and $\bar{p}_t$ respectively. Hence we have $p_0 + \bar{p}_0 = 1$, and $p_t + \bar{p}_t = 1$. Similarly, the components of M may be read off from

$$\begin{aligned}\mathbf{x}^T M \mathbf{x} =& 2p_0\xi_1^2 + 2\left(2p(y,y) + D(y,y|n,y) + D(n,y|y,y)\right)\xi_1\xi_2 \\ &+ 2p_t\xi_2^2 + 2\left(D(y,y|n,y) - D(n,y|y,y)\right)\mu_1\xi_2 \end{aligned} \tag{3.32}$$

Again the properties (3.22), (3.23) have been used.

For large N,

$$(1+z)^N \approx e^{N(z-\frac{1}{2}z^2)} \tag{3.33}$$

hence the decoherence functional now has the form

$$\begin{aligned}D(\bar{n}_1, \bar{n}_2|\bar{n}_1', \bar{n}_2) = \int d\mu_1 d\xi_1 d\xi_2 \ \exp\big(& i\mu_1(\bar{n}_1 - \bar{n}_1') \\ & -i\xi_1(\bar{n}_1 + \bar{n}_1' - 2Np_0) - i\xi_2(2\bar{n}_2 - 2Np_t) - \mathbf{x}^T A \mathbf{x}\big)\end{aligned} \tag{3.34}$$

where here, and in what follows, we will drop overall normalization factors (these are readily recovered if required). A is a symmetric 3×3 matrix, defined by

$$\mathbf{x}^T A \mathbf{x} = N\mathbf{x}^T M \mathbf{x} - \frac{1}{2}N(\mathbf{x}\cdot\mathbf{v})^2 \tag{3.35}$$

Its explicit components, which will be important below, are

$$A_{00} = 0, \quad A_{01} = 0 \tag{3.36}$$

$$A_{02} = iN \ \mathrm{Im}\left(D(y,y|n,y)\right) \tag{3.37}$$

$$A_{11} = 2Np_0\bar{p}_0 \tag{3.38}$$

$$A_{12} = 2N\left[p(y,y) - p_0p_t + \mathrm{Re}\left(D(y,y|n,y)\right)\right] \tag{3.39}$$

$$A_{22} = 2Np_t\bar{p}_t \tag{3.40}$$

The integrals may then be carried out, with the result,

$$\begin{aligned}D(\bar{n}_1, \bar{n}_2|\bar{n}_1', \bar{n}_2) = \exp\Big(& -\alpha(\bar{n}_1 - \bar{n}_1')^2 - \beta(\bar{n}_1 + \bar{n}_1' - 2Np(y_0))^2 \\ & -i\gamma(\bar{n}_1 - \bar{n}_1')(\bar{n}_2 - Np_t) \\ & -i\nu(\bar{n}_1 - \bar{n}_1')(\bar{n}_1 + \bar{n}_1' - 2Np_0)\Big)\end{aligned} \tag{3.41}$$

where the real coefficients $\alpha, \beta, \gamma, \nu$ are given by

$$\alpha = \frac{A_{11}A_{22} - A_{12}^2}{4A_{11}(iA_{02})^2} \tag{3.42}$$

$$\beta = \frac{1}{4A_{11}} \tag{3.43}$$

$$\gamma = \frac{1}{iA_{02}} \tag{3.44}$$

$$\nu = \frac{A_{12}}{2A_{11}(iA_{02})} \tag{3.45}$$

This is now the decoherence functional for precisely specified values of $\bar{n}$.

To complete the calcuation of the decoherence functional a further coarse-graining over $\bar{n}$ is required, corresponding to imprecise specification of $\bar{n}$. This involves summing $\bar{n}_1$, $\bar{n}'_1$ and $\bar{n}_2$ over ranges of values, and is most easily achieved, at least approximately, by taking $\bar{n}$ to be continuous and integrating with a Gaussian smearing. Denoting the coarse-grained values of $\bar{n}$ by $\bar{N}$, we have

$$D(\bar{N}_1, \bar{N}_2|\bar{N}'_1, \bar{N}_2) = \int d\bar{n}_1 \; d\bar{n}'_1 \; d\bar{n}_2 \; D(\bar{n}_1, \bar{n}_2|\bar{n}'_1, \bar{n}_2) \\ \times \frac{1}{(2\pi\sigma^2)^{3/2}} \; \exp\left(-\frac{(\bar{n}_1 - \bar{N}_1)^2}{2\sigma^2} - \frac{(\bar{n}'_1 - \bar{N}'_1)^2}{2\sigma^2} - \frac{(\bar{n}_2 - \bar{N}_2)^2}{2\sigma^2} \right) \tag{3.46}$$

where the ranges of integration are $-\infty$ to $+\infty$. The coarse-grained variables $\bar{N}$ are understood to have significance only to up order σ.

The integrals are most easily carried out by changing variables to $\bar{n}_1 + \bar{n}'_1$ and $\bar{n}_1 - \bar{n}'_1$, and by making use of the formula

$$\int dx \; \exp\left(-a(x - x_1)^2 - b(x - x_2)^2 + icx \right) \\ = \left(\frac{\pi}{(a+b)} \right)^{\frac{1}{2}} \; \exp\left(-\frac{c^2}{4(a+b)} - \frac{ab}{(a+b)}(x_1 - x_2)^2 + i\frac{c}{(a+b)}(ax_1 + bx_2) \right) \tag{3.47}$$

One thus obtains

$$D(\bar{N}_1, \bar{N}_2|\bar{N}'_1, \bar{N}_2) \\ = \exp\big(\; -\tilde{\alpha}(\bar{N}_1 - \bar{N}'_1)^2 - \tilde{\beta}(\bar{N}_1 + \bar{N}'_1 - 2Np_0)^2 \\ -\tilde{\epsilon}(\bar{N}_2 - Np_t)^2 - \tilde{\phi}(\bar{N}_2 - Np_t)(\bar{N}_1 + \bar{N}'_1 - 2Np_0) \\ -i\tilde{\gamma}(\bar{N}_1 - \bar{N}'_1)(\bar{N}_2 - Np_t) - i\tilde{\nu}(\bar{N}_1 - \bar{N}'_1)(\bar{N}_1 + \bar{N}'_1 - 2Np_0) \big) \tag{3.48}$$

where

$$\tilde{\alpha} = \frac{b}{(1 + 4\sigma^2 b)}, \quad b = \alpha + \frac{\gamma^2\sigma^2}{2} + \frac{\nu^2\sigma^2}{(1 + 4\sigma^2\beta)} \tag{3.49}$$

$$\tilde{\beta} = \frac{\beta}{(1+4\sigma^2\beta)} + \frac{\sigma^2\nu^2}{(1+4\sigma^2 b)(1+4\sigma^2\beta)^2} \tag{3.50}$$

$$\tilde{\epsilon} = \frac{\gamma^2\sigma^2}{(1+4\sigma^2 b)} \tag{3.51}$$

$$\tilde{\phi} = \frac{2\sigma^2\nu\gamma}{(1+4\sigma^2 b)(1+4\sigma^2\beta)} \tag{3.52}$$

$$\tilde{\gamma} = \frac{\gamma}{(1+4\sigma^2 b)} \tag{3.53}$$

$$\tilde{\nu} = \frac{\nu}{(1+4\sigma^2 b)(1+4\sigma^2\beta)} \tag{3.54}$$

Eq.(3.48) is the decoherence functional for densities of the "yes" alternative, coarse-grained to a width σ, and for histories characterized by projections at two moments of time.

3(D). The Degree of Decoherence

A reasonable measure of approximate decoherence is the size of the off-diagonal terms in comparison to the probabilities, Eq.(1.8). Denoting the diagonal parts of the decoherence functional by $p(\bar{N}_1, \bar{N}_2)$, we have,

$$\frac{|D(\bar{N}_1, \bar{N}_2|\bar{N}_1', \bar{N}_2)|}{p(\bar{N}_1, \bar{N}_2)^{\frac{1}{2}} p(\bar{N}_1', \bar{N}_2)^{\frac{1}{2}}} = \exp\left(-(\tilde{\alpha}-\tilde{\beta})(\bar{N}_1 - \bar{N}_1')^2\right) \tag{3.55}$$

hence the degree of approximate decoherence is controled by $\tilde{\alpha}-\tilde{\beta}$. Since the coarse-grained density $\bar{N}$ has significance only up to order σ, the degree of decoherence, which we denote ϵ, is at worst given by

$$\epsilon = \exp\left(-(\tilde{\alpha}-\tilde{\beta})\sigma^2\right) \tag{3.56}$$

For small σ, $\tilde{\alpha} \approx \alpha$ and $\tilde{\beta} \approx \beta$. We will see in the next section that, for the models we are interested in, the probabilities p_0, p_t, *etc.* for the component systems are of order 1, whilst the off-diagonal terms of the decoherence functional are much smaller. ¿From Eqs.(3.46)–(3.40), this means that $\alpha >> \beta$, the important term in α is the term A_{02}, and the terms A_{11}, A_{22} and A_{12} are important only in their N-dependence. It is also convenient to write the coarse-graining parameter σ as a fraction f of the total particle number N, so $\sigma = fN$. The degree of decoherence

is therefore of the form

$$\epsilon \approx \exp\left(-\frac{N\Gamma f^2}{(\mathrm{Im}D(y,y|n,y))^2}\right) \tag{3.57}$$

where

$$\Gamma = p_t\bar{p}_t - \frac{(p(y,y) - p_0p_t + \mathrm{Re}D(y,y|n,y))^2}{p_0\bar{p}_0} \tag{3.58}$$

This is the expected result, and the main technical result of this paper. The degree of decoherence improves with increasing N. It also improves as the degree of decoherence of the component systems improves (*i.e.*, as $|D(y,y|n,y)|$ gets smaller). Moreover, decoherence also relies on the factor f not being too small. This means that the density $\bar{n}$ must be partitioned into macroscopically distinct sets for there to be sufficient decoherence.

Note that Eq.(3.57) invites one to use the ratio of $(\mathrm{Im}D(y,y|n,y))^2$ to Γ as a measure of approximate decoherence of the component systems (rather than, for example, the condition (1.8)) – not an immediatey obvious measure. Having said that, it will in fact turn out that for the model of the next section, the probabilities are of order 1, so Γ is of order 1. (See figure 3.)

Note also that the decoherence functional (3.48) involves only the imaginary part of the decoherence functional of the component systems. As we shall see in the next section, it is possible to have $\mathrm{Re}D(y,y|n,y) = 0$ but $\mathrm{Im}D(y,y|n,y) \neq 0$. This means that the component systems could be exactly consistent but the total system not exactly consistent. There is no contradiction since the decoherence functional for the total system is a sum of products of the decoherence functionals for the component systems, so $\mathrm{Re}D(y,y|n,y) = 0$ does not imply that $\mathrm{Re}D(\bar{N}_1,\bar{N}_2|\bar{N}_1',\bar{N}_2)$ is exactly diagonal. Exact *decoherence* of the component systems (rather than just consistency), however, does imply exact decoherence of the total system. The significance of this, if anything, is to underscore decoherence, Eq.(1.7), as a physically more meaningful condition than consistency, Eq.(1.6).

Given decoherence we may now examine the probabilities for the histories. These are given by

$$p(\bar{N}_1,\bar{N}_2) = \exp\Big(-4\tilde{\beta}(\bar{N}_1 - Np_0)^2 - \tilde{\epsilon}(\bar{N}_2 - Np_t)^2 \\ -2\tilde{\phi}(\bar{N}_2 - Np_t)(\bar{N}_1 - Np_0)\Big) \tag{3.59}$$

This means that the probabilities for $\bar{N}_1$ only, or for $\bar{N}_2$ only, are peaked about Np_0 or Np_t, as one might expect. Beyond this, the form of the probabilitiy distribution (3.59) is not of much significance. The model is too simple for us to expect approximate evolution equations for $\bar{N}$.

There is nothing in this section that refers to the notion of conservation. The above result is essentially a combinatoric one. As we will see in the next section, the role of conservation is to ensure that $|D(y, y|n, y)|$ is small, and moreover, becomes smaller as the coarse-graining volume increases.

4. THE SPIN SYSTEM MODEL

We now describe a particular model which is of the type discussed in Section III, and in which we expect decoherence through approximate conservation.

4(A). The Model

The model consists of a chain of very large number M of locally coupled "atoms", each of which can be in one of two states, call them spin up and spin down. (We follow Ref.[20]). The spins interact via the Hamiltonian

$$H = -\frac{\chi}{2} \sum_{n=1}^{M} \overrightarrow{\sigma}_n \cdot \overrightarrow{\sigma}_{n+1} \tag{4.1}$$

where $\overrightarrow{\sigma}_n$ is the 3-vector whose components are the Pauli matrices. We impose periodic boundary conditions, $\overrightarrow{\sigma}_{N+1} = \overrightarrow{\sigma}_1$. Up to an additive constant, the Hamiltonian may be written

$$H = -\chi \sum_{n=1}^{M} p^{n,n+1} \tag{4.2}$$

Here, $p^{n,n+1}$ is a spin exchange operator. It has the effect of leaving aligned spins alone, and interchanging oppositely aligned spins. For example,

$$p^{1,2} \, |\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle \tag{4.3}$$

The ground state is the state with all spins pointing up and has eigenvalue zero. The next excited states consist of the M states for which one spin is down and the rest up, and we denote them $|\phi_k\rangle$, where

$$|\phi_1\rangle = |\downarrow\uparrow\uparrow\cdots\uparrow\rangle \tag{4.4}$$

$$|\phi_2\rangle = |\uparrow\downarrow\uparrow\cdots\uparrow\rangle \tag{4.5}$$

and so on, *i.e.*, $|\phi_k\rangle$ denotes the state in which spin number k is down and the rest are up. These are the so-called spin waves [20]. Then it is readily seen that

$$p^{k,k+1} \, |\phi_k\rangle = |\phi_{k+1}\rangle \tag{4.6}$$

$$p^{k,k+1} \, |\phi_{k+1}\rangle = |\phi_k\rangle \tag{4.7}$$

and

$$p^{k,k+1}\ |\phi_n\rangle = 0 \tag{4.8}$$

if $n < k$ or $n > k+1$.

The states $|\phi_k\rangle$ are not eigenstates of the Hamiltonian, but the combinations

$$|\psi_\ell\rangle = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \exp\left(\frac{2\pi i \ell k}{M}\right)\ |\phi_k\rangle \tag{4.9}$$

are, with eigenvalues $E_\ell = -2\chi\ \cos\left(\frac{2\pi\ell}{M}\right)$, and $\ell = 1, \cdots M$. Their normalization follows from the identity,

$$\frac{1}{M} \sum_{k=1}^{M} \exp\left(\frac{2\pi i k(n-m)}{M}\right) = \delta_{nm} \tag{4.10}$$

where δ_{nm} is the Kronecker delta.

4(B). Coarse Grainings

There are two conserved quantities for this model: the total energy and the total spin. We will study coarse-grainings which ask for the total spin in a region of the chain. We take the crudest coarse-graining, which is to divide the chain into two regions, region 1 and region 2, of size M_1 and M_2, where $M_1 + M_2 = M$. We will construct projections which ask for the total number of down spins in region 1.

The model so far is not yet exactly of the type described in Section III because there, a non-interacting Hamiltonian was assumed, whereas here, we have an interacting one. To get around this, and hence to use the results of Section III, we do the following. First, we restrict attention to the subspace in which there is a fixed number N of spins pointing down. This subspace is invariant under Hamiltonian evolution, so the set of N downward pointing spins may be regarded as a set of N interacting "particles". Second, and more importantly, we assume that $1 << N << M$. This is a kind of "dilute gas" assumption, and we expect it to allow us to neglect the interaction between downward pointing spins.

More precisely, we replace the N particle Hilbert space $\mathcal{H}_N$ with a tensor product of N Hilbert spaces $\mathcal{H}_1$, where $\mathcal{H}_1$ denotes the Hilbert space of states with

one spin pointing down described above. We also take the Hamiltonian H_N for the N particles to be of the form

$$\begin{aligned} H_N = & H_1 \otimes 1 \otimes 1 \otimes \cdots \\ & + 1 \otimes H_1 \otimes 1 \otimes \cdots \\ & + \cdots \end{aligned} \tag{4.11}$$

where H_1 is the Hamiltonian for the states with one spin pointing down, operating on $\mathcal{H}_1$. The Hamiltonian is thus of the form required for the results of Section III to be applicable.

Clearly some work is required to fully justify this approximation. The Hilbert space $\mathcal{H}_N$ does not have exactly the same dimension is the tensor product of the N spaces $\mathcal{H}_1$, although they are close for sufficiently large N and M. Also, one would expect the effective Hamiltonian to include a term in the Hamiltonian preventing two down spins from occupying the same site, and a term describing the interacting between neighbouring down spins. What we are assuming, in effect, is that these extra terms can be neglected if the gas of downward pointing spins is sufficiently dilute.

Even if this approximation cannot be justified, we could always postulate a model of the above type. This would admittedly be unphysical, but at least it provides a framework in which we can investigate the mathematical properties we are interested in, which is the main aim of this paper.

The projection $P_{\bar{n}}$ onto the total spin in region 1, is constructed from the projection for the individual particles, as described in Section III, Eq.(3.5). For each individual particle we introduce the projections,

$$P = \sum_{k=1}^{M_1} |\phi_k\rangle\langle\phi_k| \tag{4.12}$$

$$\bar{P} = \sum_{k=M_1+1}^{M} |\phi_k\rangle\langle\phi_k| \tag{4.13}$$

In the 1-particle subspace, P and $\bar{P}$ ask whether the downward spin is in region 1 or not in region 1 (*i.e.*, in region 2), respectively. We also denote these alternatives y and n, respectively, in accordance with the notation of Section III. As shown in Section III, the computation of the decoherence functional onto histories in which

the total spin in region 1 is specified reduces to a computation of the decoherence functional for the component systems.

4(C). The Decoherence Functional of the Component Systems

The Hamiltonian for the 1-particle subspace $\mathcal{H}_1$ may be written,

$$H_1 = \sum_{\ell=1}^{M} E_\ell \ |\psi_\ell\rangle\langle\psi_\ell| \tag{4.14}$$

and hence the unitary evolution operator is

$$e^{-iH_1 t} = \sum_{\ell=1}^{M} \ e^{-itE_\ell} \ |\psi_\ell\rangle\langle\psi_\ell| \tag{4.15}$$

The Hamiltonian may also be written,

$$H_1 = -\chi \sum_{k=1}^{M} \left(|\phi_k\rangle\langle\phi_{k-1}| + |\phi_{k-1}\rangle\langle\phi_k|\right) \tag{4.16}$$

The decoherence functional for a simple two-time history is

$$D(\alpha_1, \alpha_2|\alpha_1', \alpha_2) = \mathrm{Tr}\left(P_{\alpha_2}(t) P_{\alpha_1} \rho P_{\alpha_1'}\right) \tag{4.17}$$

where $P_{\alpha_2}(t) = e^{iH_1 t} P_{\alpha_2} e^{-iH_1 t}$, and P_α denotes P and $\bar{P}$. Take an initial state which is a superposition states in which the particles is in region 1 and 2,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\phi_{k_1}\rangle + |\phi_{k_2}\rangle\right) \tag{4.18}$$

where $1 \le k_1 \le M_1$, and $M_1 + 1 \le k_2 \le M$. Then

$$P_\alpha \ |\Psi\rangle\langle\Psi| \ P_{\alpha'} = \frac{1}{2} \ |\phi_{k_\alpha}\rangle\langle\phi_{k_{\alpha'}}| \tag{4.19}$$

where for the moment we let α take the values $1, 2$, corresponding to y, n. The decoherence functional then is

$$D(\alpha_1, \alpha_2|\alpha_1', \alpha_2) = \frac{1}{2}\langle\phi_{k_{\alpha_1}}|P_{\alpha_2}(t)|\phi_{k_{\alpha_1'}}\rangle \tag{4.20}$$

For this, we then have, for example

$$D(y, y|n, y) = \frac{1}{2}\langle\phi_{k_1}|P(t)|\phi_{k_2}\rangle \tag{4.21}$$

$$p(n, y) = \frac{1}{2}\langle\phi_{k_2}|P(t)|\phi_{k_2}\rangle \tag{4.22}$$

and so on.

Let $P_{\alpha_2} = P$. Then it is readily shown that

$$\langle \phi_n | P(t) | \phi_{n'} \rangle = \frac{1}{M} \sum_{\ell=1}^{M} \sum_{\ell'=1}^{M} e^{it(E_\ell - E_{\ell'})} \exp\left(\frac{2\pi i (n\ell - n'\ell')}{M} \right) d(\ell, \ell') \tag{4.23}$$

where

$$d(\ell, \ell') = \sum_{k=1}^{M_1} \exp\left(-\frac{2\pi i k (\ell - \ell')}{M} \right) \tag{4.24}$$

Now, writing $x = 2\pi(\ell - \ell')/M$, we have

$$d(\ell, \ell') = \sum_{k=1}^{M_1} e^{-ikx} = \frac{(e^{-iM_1 x} - 1)}{(1 - e^{ix})} \tag{4.25}$$

The properties of the decoherence functional may be understood through $d(\ell, \ell')$. Clearly, if $M_1 = M$, then $d(\ell, \ell') = M\delta_{\ell\ell'}$, and $P(t)$ would be diagonal in the $|\phi_n\rangle$'s. More generally, if M_1 and M are very large, then $d(\ell, \ell')$ is very small when $\ell \neq \ell'$, and $d(\ell, \ell) = M_1$.

Curiously, when $M_1 = M_2$ the real part of the decoherence functional vanishes. Thus, this coarse-graining exhibits exact consistency, but not exact decoherence. It seems reasonable that this sort of consistency, where the imaginary part of the decoherence functional is non-zero, will only occur due to symmetries in the system and choice of histories.

4(D). Numerical Results

The decoherence functional for the component systems may be computed numerically using Eq.(4.20). There are three parameters, t, M and M_1. The decoherence functional for various ranges of the parameters is computed below and the results plotted in figures 1–3.

4(E). Summary of Results

The results of the numerical calculation may be concisely summarized as follows. The probabilities $p(y, y)$, $p(y, n)$, *etc.* are of order 1. The off-diagonal terms

of the decoherence functional, divided by the probabilities, are typically of order $M^{-\frac{1}{2}}$, hence the degree of decoherence for the 1-particle case is of this order. This degree of decoherence is not particularly good (compared, for example, to the quantum Brownian motion models, where one typically finds that the degree of decoherence is an exponential function of the coarse-graining parameters [10]). It is no better than one would expect from the overlap of two arbitrary vectors in a Hilbert space of dimension M [21]. Physically, this is not surprising, because the histories in question differ by just one quantum of spin, and one would not expect the interference between such spins to be substantially supressed.

However, when inserted into the expression for the decoherence functional of the N-particle case, (3.48), a vastly improved degree of decoherence is obtained, for sufficiently large N. Moreover, the important feature about the decoherence functional (4.20) is that the off-diagonal terms decrease as the size M_1 of the"smearing volume" increases from zero (until M_1 reaches the value about $M/2$). Correspondingly, the off-diagonal terms of the decoherence functional of the N-particle case (3.48) decrease very rapidly as M_1 increases from zero.

5. SUMMARY AND CONCLUSIONS

This paper marks a first step towards the problem of computing the decoherence functional for hydrodynamic histories. We considered systems of essentially non-interacting particles and studied histories of projections onto densities of those particles. Our principal aim was to show, in the context of some simple models, how decoherence can come about as a result of approximate conservation, in tune with the general ideas put forward by Gell-Mann and Hartle [4,5].

The results of Section IV show explicitly how approximate decoherence is related to approximate conservation: the degree of decoherence increases as the smearing volume increases (at least, for volumes less than half the total volume of the system). The degree of decoherence, however, is not very good. This is because we considered single particle models in Section IV, and the histories differ by just one quantum of spin. The histories are therefore not "macroscopically distinct" and one would expect that interference between them could still be quite noticeable. Differently put, it is because the fluctuations in variables in question (*e.g.*, number density) are comparable to the values of the variables themselves.

To obtain adequate decoherence, it is necessary to have a large number N of particles and partition the number density by large ranges σ. It is here that the results of Section III came in. We found that when the histories do differ by macroscopically significant amounts, interference is destroyed very efficiently. Clearly what is happening here is that the fluctuations in the variables are much smaller than the variables themselves.

¿From this we conclude that, in these simple models, decoherence requires *two* distinct phenomena: approximate conservation, and large particle number partitioned into large ranges. Approximate conservation ensures that the dynamical variables projected onto become sufficiently slowly varying for sufficiently large smearing volume. Large particle number partitioned into large ranges ensures that the quantum fluctuations in the local densities are smaller than the values of the variables themselves. These conclusions concure with the general expectations expressed in Refs.[4,5].

The restriction to non-interacting particles may seem rather unrealistic. It means that conservation may already be seen at the one particle level, whilst for

an interacting theory, it is only seen for the whole collection of particles. Although physically unrealistic, it had the mathematical advantage that the effects of large particle number and approximate conservation could be cleanly seperated. Moreover, one would not expect the inclusion of interactions to substantially modify our conclusions (although this is clearly an important extension to carry out). The point is that here, it is the approximate conservation and large particle number that produce decoherence, not the interactions. This is in stark contrast to the vast majority of other models studied in the literature on decoherence, in which it is interactions (usually with another system) that are held responsible for decoherence.

This paper is, as stated, a first step in an extensive investigation, and there are therefore many ways in which it may be developed. Perhaps the next step is to seek a more sophisticated model in which one would expect the probabilities for histories to be peaked about interesting approximately deterministic evolution equations.

A technically very different but similar in spirit investigation is that of Calzetta and Hu [11], who considered the decoherence of histories characterized by the values of n-point functions for fields. They did not make any contact with the notion of approximate conservation, but it is similar to our work in that it is, to the best of our knowledge, the only other concrete calculation of decoherence which avoids a system–environment split. It would be of interest to find more detailed connections between their work and ours.

These and other questions will be the topic of future publications.

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APPENDIX A: EXACT EVALUATION OF THE DECOHERENCE FUNCTIONAL

The integrals over the λ's may be evaluated exactly, and we present here the result of that calculation.

$$\begin{aligned} D(\bar{n}_1,\bar{n}_2|\bar{n}_1',\bar{n}_2) &= \mathrm{Tr}\left(P_{\bar{n}_2}\ e^{-iHt}\ P_{\bar{n}_1}\ \rho\ P_{\bar{n}_1'}\ e^{iHt}\right) \\ &= \frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}d\lambda_1\int_{-\pi}^{\pi}d\lambda_1'\int_{-\pi}^{\pi}d\lambda_2\ e^{-2i\lambda_1\bar{n}_1+2i\lambda_1'\bar{n}_1'-2i\lambda_2\bar{n}_2} \\ &\times\Big[e^{2i(\lambda_1+\lambda_2-\lambda_1')}p(y,y)+e^{2i\lambda_2}p(n,y) \\ &\quad +e^{2i(\lambda_1-\lambda_1')}p(y,n)+p(n,n) \\ &\quad\quad +e^{2i(\lambda_1+\lambda_2)}D(y,y|n,y)+e^{2i(\lambda_2-\lambda_1')}D(n,y|y,y) \\ &\quad\quad\quad +e^{2i\lambda_1}D(y,n|n,n)+e^{-2i\lambda_1'}D(n,n|y,n)\Big]^N \end{aligned} \tag{A1}$$

Now note that the term in brackets raised to the power N may be written

$$\left[A+Be^{2i\lambda_2}\right]^N=\sum_{k=0}^{N}\binom{N}{k}\ A^{N-k}\ B^k\ e^{2i\lambda_2 k} \tag{A2}$$

using the binomial expansion, where

$$A=e^{2i(\lambda_1-\lambda_1')}p(y,n)+p(n,n)+e^{2i\lambda_1}D(y,n|n,n)+e^{-2i\lambda_1'}D(n,n|y,n) \tag{A3}$$

$$B=e^{2i(\lambda_1-\lambda_1')}p(y,y)+p(n,y)+e^{2i\lambda_1}D(y,y|n,y)+e^{-2i\lambda_1'}D(n,y|y,y) \tag{A4}$$

The integral over λ_2 may now be evaluated, with the result,

$$D(\bar{n}_1,\bar{n}_2|\bar{n}_1',\bar{n}_2)=\frac{1}{(2\pi)^2}\int d\lambda_1 d\lambda_1'\ e^{-2i\lambda_1\bar{n}_1+2i\lambda_1'\bar{n}_1'}\ \binom{N}{\bar{n}_2}\ A^{N-\bar{n}_2}\ B^{\bar{n}_2} \tag{A5}$$

We now use the binomial expansion again on $A^{N-\bar{n}_2}$ and $B^{\bar{n}_2}$. Write

$$A=e^{-2i\lambda_1'}C+D \tag{A6}$$

$$B=e^{-2i\lambda_1'}E+F \tag{A7}$$

where

$$C=e^{2i\lambda_1}p(y,n)+D(n,n|y,n) \tag{A8}$$

$$D=p(n,n)+e^{2i\lambda_1}D(y,n|n,n) \tag{A9}$$

$$E=e^{2i\lambda_1}p(y,y)+D(n,y|y,y) \tag{A10}$$

$$F=p(n,y)+e^{2i\lambda_1}D(y,y|n,y) \tag{A11}$$

Expanding $A^{N-\bar{n}_2}$ and $B^{\bar{n}_2}$, the integral over λ_1' may then be carried out, with the result,

$$D(\bar{n}_1,\bar{n}_2|\bar{n}_1',\bar{n}_2) = \frac{1}{2\pi}\binom{N}{\bar{n}_2}\int d\lambda_1\; e^{-2i\lambda_1\bar{n}_1}\sum_{k=0}^{N-\bar{n}_2}\sum_{j=0}^{\bar{n}_2}\binom{N-\bar{n}_2}{k}\binom{\bar{n}_2}{j} \times\; C^k\; D^{N-\bar{n}_2-k}\; E^j\; F^{\bar{n}_2-j}\;\delta_{k+j,\bar{n}_1'} \qquad (A12)$$

Repeating the binomial expansion of C^k *etc.* on final time, the last integration over λ_1 may be performed, and we arrive at the result,

$$\begin{aligned} D(\bar{n}_1,\bar{n}_2|\bar{n}_1',\bar{n}_2) = \binom{N}{\bar{n}_2} & \sum_{k=0}^{N-\bar{n}_2}\sum_{j=0}^{\bar{n}_2}\sum_{\ell=0}^{k}\sum_{m=0}^{N-\bar{n}_2-k}\sum_{r=0}^{j}\sum_{s=0}^{\bar{n}_2-j} \\ & \times\; \binom{N-\bar{n}_2}{k}\binom{\bar{n}_2}{j}\;\delta_{k+j,\bar{n}_1'}\;\delta_{\ell+m+r+s,\bar{n}_1} \\ & \times\; \binom{k}{\ell}[D(n,n|y,n)]^{k-\ell}\;[p(y,n)]^{\ell} \\ & \times\; \binom{N-\bar{n}_2-k}{m}[p(n,n)]^{N-\bar{n}_2-k-m}\;[D(y,n|n,n)]^{m} \\ & \times\; \binom{j}{r}[D(n,y|y,y)]^{j-r}\;[p(y,y)]^{r} \\ & \times\; \binom{\bar{n}_2-j}{s}[p(n,y)]^{\bar{n}_2-j-s}\;[D(y,y|n,y)]^{s} \qquad (A13) \end{aligned}$$

For reasonably small N, this expression can be evaluated numerically. For large N (and $\bar{n}_1$, $\bar{n}_1'$, $\bar{n}_2$), the sums can be estimated by replacing them by integrals, and assuming that the dominant contributions come from the middle of the ranges, *i.e.*, from values of j,k,ℓ,m,r,s that are large (although we have not carried this out explicitly).

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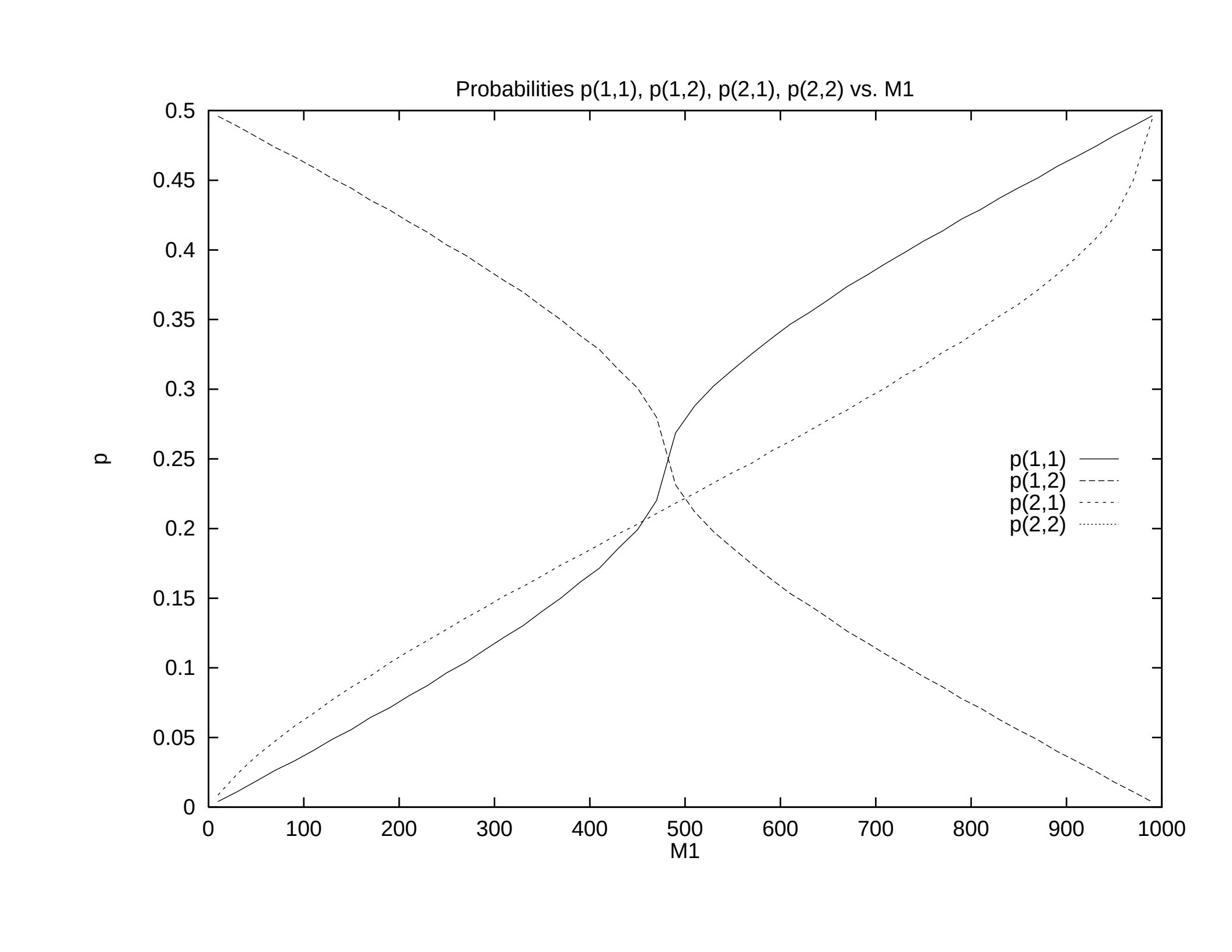
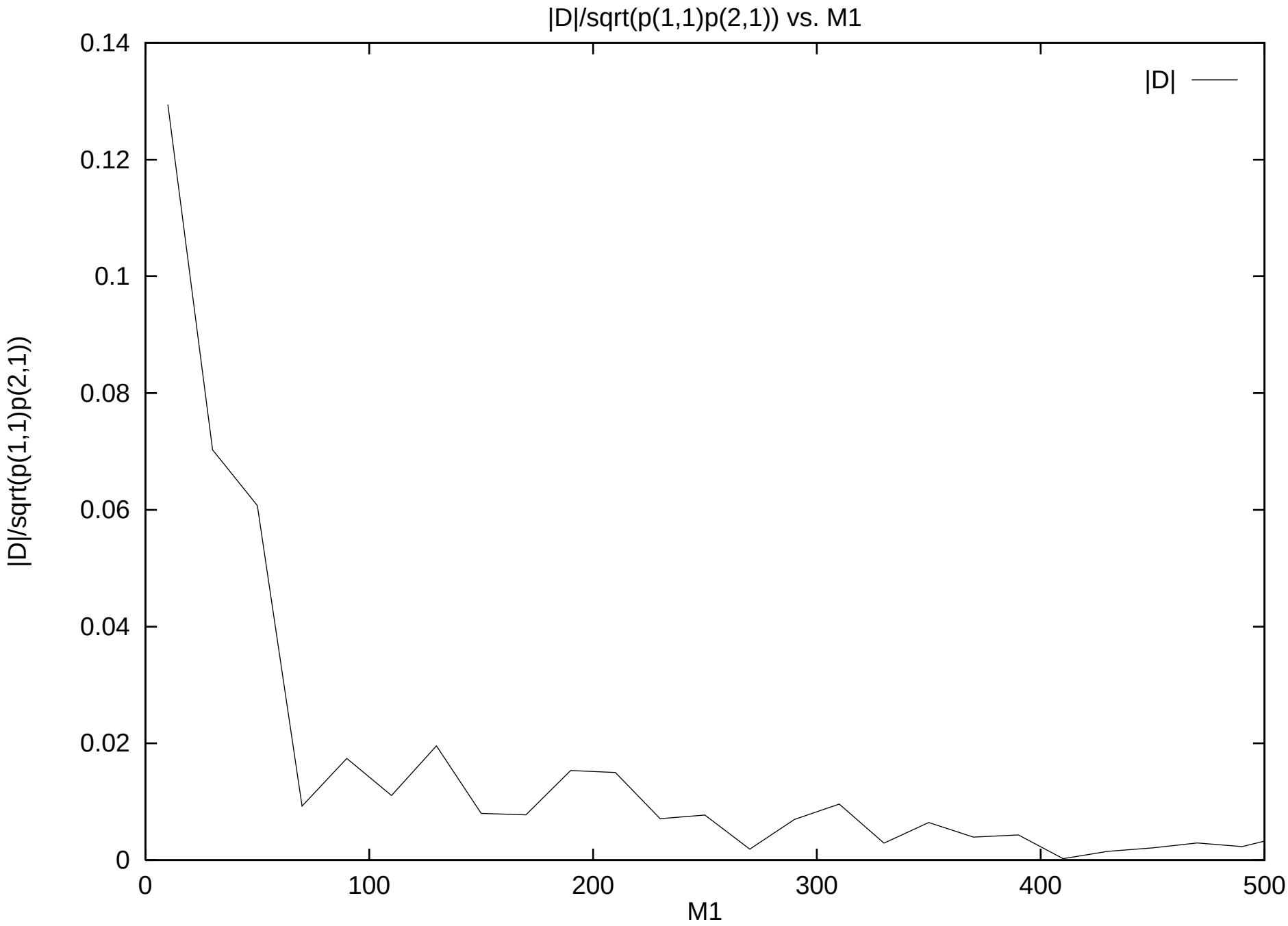
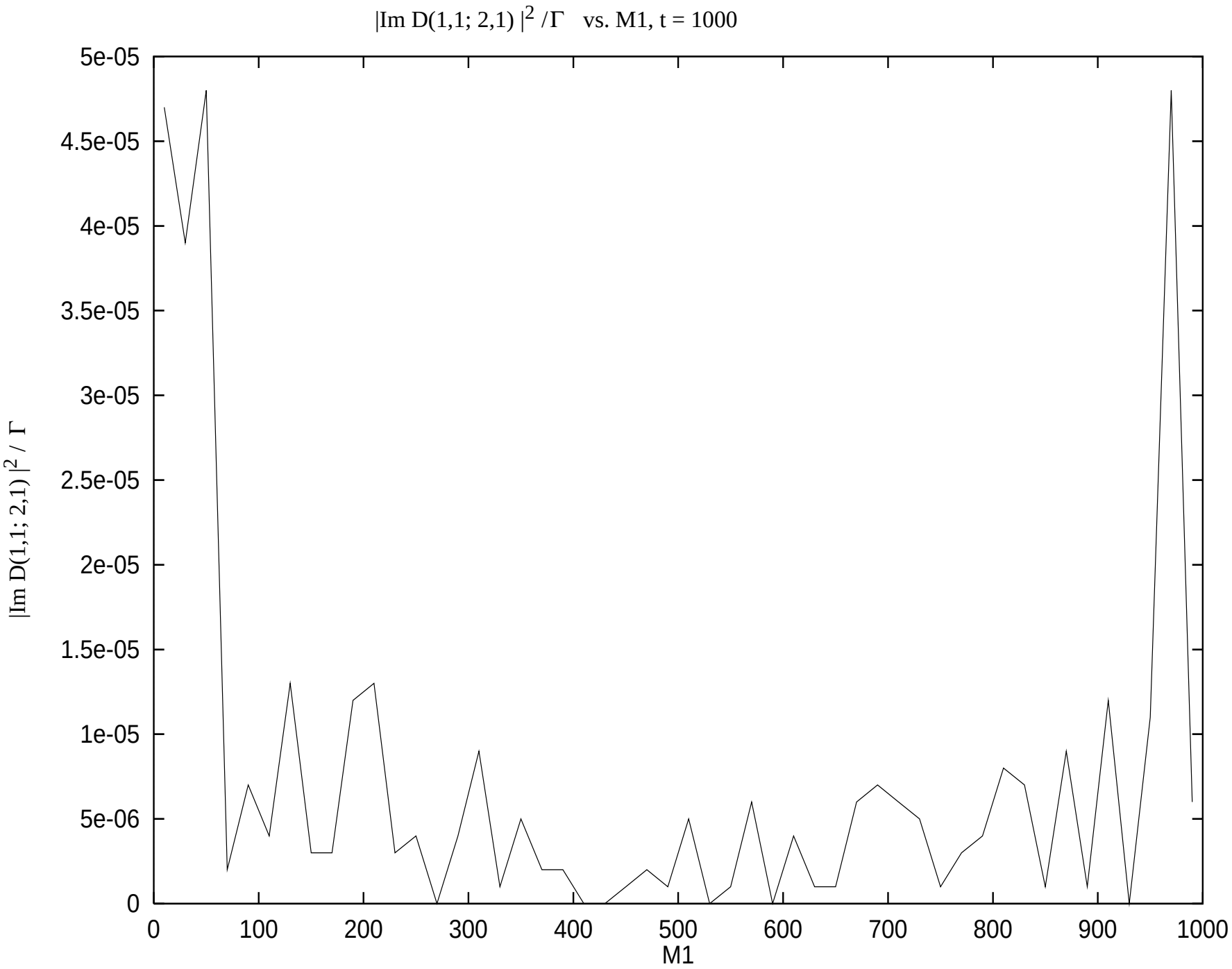
Figure 1. The probabilities $p(1,1)$, $p(1,2)$, $p(2,1)$, $p(2,2)$ as a function of the coarse-graining M_1. This plot was produced at $t = 1000$ for a string of $M = 1000$ spins; the energy constant is $\chi = 1$. Note that for M_1 close to 500 (half the spins), the probabilities are all close to 0.25.

Figure 2. $|D[1,1;2,1]|/\sqrt{p(1,1)p(2,1)}$ vs. M_1. For this plot the parameters are $t = 1000$, $M = 1000$, $\chi = 1$. Note this quantitity is highly peaked for low values of M_1, indicating that better decoherence results from coarser graining.

Figure 3. $|\mathrm{Im}D[1,1;2,1]|^2/\Gamma$ vs. M_1. For this plot the parameters are $t = 1000$, $M = 1000$, $\chi = 1$. This also exhibits peaking for low values of M_1, indicating good decoherence for coarse graining. Note also that the absolute value of this quantity is low; this will produce very good decoherence for many-spin systems, as shown in equation (3.57).


Probabilities p(1,1), p(1,2), p(2,1), p(2,2) vs. M1
p
M1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0
100
200
300
400
500
600
700
800
900
1000
p(1,1)
p(1,2)
p(2,1)
p(2,2)


|D|/sqrt(p(1,1)p(2,1)) vs. M1
|D|
|D|/sqrt(p(1,1)p(2,1))
M1
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
100
200
300
400
500


|Im D(1,1; 2,1) |2 /Γ vs. M1, t = 1000
|Im D(1,1; 2,1) |2 / Γ
M1
5e-05
4.5e-05
4e-05
3.5e-05
3e-05
2.5e-05
2e-05
1.5e-05
1e-05
5e-06
0
0
100
200
300
400
500
600
700
800
900
1000